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Homological finiteness properties of pro-p modules over metabelian pro-p groups ☆

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Abstract

We characterize the modules B of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$, where G is a topologically finitely generated metabelian pro-p group that is an extension of A by Q, with A and Q abelian, and B is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module that is viewed as a pro-p $\mathbb{Z}_p[\![G]\!]$-module via the projection $G \to Q$. The characterization is given in terms of the invariant introduced by King [J.D. King, A geometric invariant for metabelian pro-p groups, J. London Math. Soc. (2) 60 (1) (1999) 83–94] and is a generalization of the case when $B = \mathbb{Z}_p$ is considered as a trivial $\mathbb{Z}_p[\![G]\!]$-module that gives the classification of metabelian pro-p groups of type FP_m [D.H. Kochloukova, Metabelian pro-p groups of type FP_m, J. Group Theory 3 (4) (2000) 419–431].

Keywords: Metabelian pro-p groups; Pro-p modules; Homological type FP_m

1. Introduction

Let G be a pro-p group. A pro-p $\mathbb{Z}_p[\![G]\!]$-module B is of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$ if B has a $\mathbb{Z}_p[\![G]\!]$-projective resolution where all modules in dimensions less or equal to m are finitely generated.

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In this paper we give a positive answer for the pro-p version of the generalized FP_m-Conjecture that classifies the homological type of some special pro-p modules over finitely generated metabelian pro-p groups (finitely generated pro-p groups means topologically finitely generated pro-p groups). We are interested in pro-p modules B over the completed group algebra $\mathbb{Z}_p[\![G]\!]$, where G is a finitely generated pro-p group that is an extension of A by Q, with A and Q abelian, and B is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module that is viewed as a $\mathbb{Z}_p[\![G]\!]$-module via the projection $G \to Q$. The classification is given in terms of the invariant Δ defined by King in [6].

The problem was first suggested for the case of modules over finitely generated metabelian abstract groups [7, Conjecture 7]. This case is proved only for $m = 1$ [7] and $m = 2$ [10] with G being a split extension of abelian groups, and it is a generalization of the FP_m-Conjecture [2]. The same problem was also formulated and proved for modules over metabelian Lie algebras which are split extension of an abelian ideal by an abelian Lie subalgebra [9]. Our main result is the proof of the pro-p case without any restriction on the type of extension.

Theorem 1.1. *Let $1 \to A \to G \to Q \to 1$ be an exact sequence of pro-p groups such that A and Q are abelian and G is finitely generated. Let B be a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module. Then the following are equivalent*:

(i) *B is of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$, where the action of G is defined via the epimorphism $G \to Q$*;
(ii) *$B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigotimes}^m_{\mathbb{Z}_p} A)$ is finitely generated as a pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action*;
(iii) *$B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is finitely generated as a pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action*;
(iv) *if $v_1 \in \Delta_B(Q)$ and $v_2, \ldots, v_{m+1} \in \Delta_A(Q)$ are such that $v_1 v_2 \cdots v_{m+1} = 1$, then $v_1 = v_2 = \cdots = v_{m+1} = 1$.*

Here, $\Delta_*(Q)$ stands for the King invariant associated to a finitely generated $\mathbb{Z}_p[\![Q]\!]$-module (see Section 2.2). The case where B is zero is degenerated and we shall tacitly exclude it in this paper.

A pro-p group G is of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$ if $\mathbb{Z}_p$ considered as a trivial pro-p $\mathbb{Z}_p[\![G]\!]$-module is of type FP_m. Note that in Theorem 1.1, the case $B = \mathbb{Z}_p$ is exactly the classification of the metabelian pro-p groups of type FP_m suggested by King in [6] and proved by Kochloukova in [8, Theorem D]. This particular case is the pro-p version of the FP_m-Conjecture. Although our result is more general than [8, Theorem D], it is worthwhile to mention that in our proof we use the fact that the result holds for $B = \mathbb{Z}_p$.

Note also that the equivalence of (i) and (iv) in Theorem 1.1 implies that, whenever m elements of $\Delta_A(Q)$ have trivial product, each one is trivial (i.e., A is m-tame). Thus in view of [8, Theorem D] we can state the following result.

Corollary 1.2. *Let $1 \to A \to G \to Q \to 1$ be an exact sequence of pro-p groups such that A and Q are abelian and G is finitely generated. Let B be a finitely generated pro-p*

$\mathbb{Z}_p[\![Q]\!]$-module that is viewed as a $\mathbb{Z}_p[\![G]\!]$-module via the projection $G \to Q$. If B is of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$, then G is of homological type FP_m over $\mathbb{Z}_p[\![G]\!]$.

The proof of Theorem 1.1 will be done in several steps. In one of them (see Theorem 3.3) we obtain the following auxiliar equivalence: B is of type FP_m over $\mathbb{Z}_p[\![G]\!]$ if and only if $\operatorname{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$ is finitely generated as a pro-p $\mathbb{F}_p[\![Q]\!]$-module for all $i \leqslant m$. This result is an important tool and it is a straightforward generalization of the case $B = \mathbb{Z}_p$ [5, Theorem 3.2], where $\operatorname{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(\mathbb{Z}_p, \mathbb{F}_p) = H_i(A, \mathbb{F}_p)$ is the ith homology group of A with coefficients in $\mathbb{F}_p$. In another step we apply the universal coefficient theorem to obtain a relationship between $\operatorname{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$ and $H_i(A, \mathbb{Z}_p)$. This relation and some results about the structure module of $H_i(A, \mathbb{Z}_p)$ proved in [5,8], will able us to verify that (i) implies (iii) and that (ii) implies (i) (see Theorems 4.1 and 4.6).

The paper is organized as follows. In Section 2 we present some basic definitions, the results of [5,8] that we will use and we discuss with details the King invariant. In Section 3 we prove the aforementioned auxiliar equivalence. Finally, Section 4 is devoted to the proof of the main result.

2. Preliminaries

2.1. Pro-p modules of type FP_m

Let G be a pro-p group, $\mathbb{Z}_p[\![G]\!]$ be the completed group algebra of G over the ring of p-adic integers $\mathbb{Z}_p$ and B be a (right) pro-p $\mathbb{Z}_p[\![G]\!]$-module. A (*right*) *$\mathbb{Z}_p[\![G]\!]$-projective resolution of* B is an exact sequence in the category of pro-p $\mathbb{Z}_p[\![G]\!]$-modules

$$\mathcal{F}: \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to B \to 0$$

where each F_i is a projective (right) pro-p $\mathbb{Z}_p[\![G]\!]$-module. Given $0 \leqslant m \leqslant \infty$, we say that *$B$ is of type FP_m over $\mathbb{Z}_p[\![G]\!]$* if B has a $\mathbb{Z}_p[\![G]\!]$-projective resolution with each F_i finitely generated (topologically or abstractly is the same [14, Lemma 7.2.2]) as a $\mathbb{Z}_p[\![G]\!]$-module for $0 \leqslant i \leqslant m$ (for all i if $m = \infty$).

Note that B is of type FP_0 over $\mathbb{Z}_p[\![G]\!]$ if and only if B is a finitely generated pro-p $\mathbb{Z}_p[\![G]\!]$-module. Furthermore, B is finitely presented pro-p $\mathbb{Z}_p[\![G]\!]$-module if and only if there is an exact sequence $R \to F_0 \to B \to 0$ of pro-p $\mathbb{Z}_p[\![G]\!]$-modules such that F_0 is free and F_0 and R are finitely generated. Then B is of type FP_1 over $\mathbb{Z}_p[\![G]\!]$ if and only if B is finitely presented.

Now let L be a (left) pro-p $\mathbb{Z}_p[\![G]\!]$-module. Given a (right) $\mathbb{Z}_p[\![G]\!]$-projective resolution $\mathcal{F}$ of B, we obtain a complex

$$\mathcal{F}_B \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} L: \cdots \to F_i \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} L \to F_{i-1} \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} L \to \cdots \to F_0 \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} L \to 0.$$

Then $\operatorname{Tor}_i^{\mathbb{Z}_p[\![G]\!]}(B, L)$ is the ith homology group $H_i(\mathcal{F}_B \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} L)$ of this complex. Good references about homological properties of profinite modules are [12,14].

Our study of pro-p modules of type FP_m is related with the module structure of the homology groups

$$H_i(A, \mathbb{F}_p) = \operatorname{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(\mathbb{Z}_p, \mathbb{F}_p),$$

where A is an abelian pro-p group and $\mathbb{Z}_p$ and $\mathbb{F}_p$ are considered as trivial $\mathbb{Z}_p[\![A]\!]$-modules. We state below some results that suit our purposes. The first one is proved in [5] and some observations about the naturalness of the morphisms can be found in [8, Lemma 3].

Theorem 2.1. [5, Theorem 3.5] *Let A be an abelian pro-p group and $\mathbb{F}_p$ be considered as trivial pro-p $\mathbb{Z}_p[\![A]\!]$-module. Then, for each i, there is a natural monomorphism*

$$\beta : \hat{\bigwedge}^i_{\mathbb{F}_p} (A \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p) \to H_i(A, \mathbb{F}_p)$$

from the completed ith exterior power to the ith homology group. If A is torsion-free then β is an isomorphism. Moreover, if a pro-p group Q acts on A, then β is a monomorphism of Q-modules, where Q acts diagonally on $\hat{\bigwedge}^i_{\mathbb{F}_p}(A \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p)$.

In the next result $\hat{S}^k_{\mathbb{Z}_p}(A)$ denotes the completed kth symmetric tensor power of A over $\mathbb{Z}_p$. Theorem 2.2 is a corollary of a result of Cartan [4].

Theorem 2.2. [8, Theorem B] *If A is an abelian pro-p group of exponent p, then $H_n(A, \mathbb{F}_p)$ has a natural filtration of $\mathbb{Z}_p$-submodules with factors that are embeddable in one of the modules*

$$\left(\hat{\bigwedge}^{n-2r}_{\mathbb{Z}_p} A\right) \hat{\otimes}_{\mathbb{Z}_p} \hat{S}^r_{\mathbb{Z}_p}(A) \quad \text{for } 0 \leqslant r \leqslant \left[\frac{n}{2}\right] \text{ and } p \neq 2,$$

$$\hat{S}^n_{\mathbb{Z}_2}(A) \quad \text{for } p = 2.$$

Furthermore, if A is a pro-p $\mathbb{Z}_p[\![Q]\!]$-module for some pro-p group Q, then the above filtration is of pro-p $\mathbb{Z}_p[\![Q]\!]$-submodules and the action of Q on the quotients is induced by the diagonal action of Q on $\hat{\bigotimes}^{n-r}_{\mathbb{Z}_p} A$ for $0 \leqslant r \leqslant [n/2]$.

Corollary 2.3. [8, Corollary C] *Let A be a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module, where Q is a pro-p group of finite rank. Then $H_n(A, \mathbb{F}_p)$ has a filtration of $\mathbb{Z}_p[\![Q]\!]$-submodules with quotients that are pro-p $\mathbb{Z}_p[\![Q]\!]$-subquotients of*

$$H_{\alpha_1}(A_1, \mathbb{F}_p) \hat{\otimes}_{\mathbb{Z}_p} H_{\alpha_2}(A_2, \mathbb{F}_p) \hat{\otimes}_{\mathbb{Z}_p} \cdots \hat{\otimes}_{\mathbb{Z}_p} H_{\alpha_s}(A_s, \mathbb{F}_p)$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_s = n$, A has a filtration

$$B_1 = A \supseteq B_2 = \operatorname{tor} A \supseteq B_3 \supseteq \cdots \supseteq B_s \supseteq B_{s+1} = 0,$$

$A_j = B_j / B_{j+1}$, A_j has exponent p for $j \geqslant 2$ and the action of Q on the above completed tensor product is the diagonal one.

We observe that the filtrations given by the above results are bounded filtrations. Furthermore, note that if M is a pro-p $\mathbb{Z}_p[\![G]\!]$-module of exponent p (possibly with $G = 1$) we can see it as a module over $\mathbb{F}_p[\![G]\!] = \mathbb{Z}_p[\![G]\!]/p\mathbb{Z}_p[\![G]\!]$ and for such modules the functor $- \hat{\otimes}_{\mathbb{F}_p[\![G]\!]} -$ coincides with $- \hat{\otimes}_{\mathbb{Z}_p}[\![G]\!] -$.

2.2. The King invariant for metabelian pro-p groups

Let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$ and $\mathbb{F}[\![T]\!]$ be the formal power series algebra with group of units $\mathbb{F}[\![T]\!]^\times$. Let Q be a finitely generated abelian pro-p group and $T(Q)$ be the set $\mathrm{Hom}(Q, \mathbb{F}[\![T]\!]^\times)$ of continuous homomorphisms from Q to $\mathbb{F}[\![T]\!]^\times$. By the universal property of $\mathbb{Z}_p[\![Q]\!]$, each $v \in T(Q)$ extends to an unique continuous ring homomorphism from $\mathbb{Z}_p[\![Q]\!]$ to $\mathbb{F}[\![T]\!]$ that we denote by $\bar{v}$.

Definition 2.4. Let Q be a finitely generated abelian pro-p group and A be a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module. The King invariant is defined by

$$\Delta_A(Q) = \{v \in T(Q) \mid \mathrm{Ann}_{\mathbb{Z}_p[\![Q]\!]}(A) \leqslant \mathrm{Ker}\,\bar{v}\} \cup \{1\}.$$

The King invariant was recently generalized for the case of pro-p groups of type FP_m in [11]. We state below some important properties of $\Delta_A(Q)$.

Proposition 2.5. [6, Corollary 2.6] *Let Q be a finitely generated abelian pro-p group, P a closed subgroup of Q and A a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module. Then A is finitely generated as a pro-p $\mathbb{Z}_p[\![P]\!]$-module if and only if $T(Q, P) \cap \Delta_A(Q) = \{1\}$, where $T(Q, P) = \{v \in T(Q) \mid v(P) = 1\}$.*

For the next property, let us fix the following notation. Let Q and Q' be finitely generated abelian pro-p groups. Then an epimorphism $\pi : Q \to Q'$ induces a monomorphism $\pi^* : T(Q') \to T(Q)$. If S_1 and S_2 are subsets of $T(Q)$, we write S_1S_2 for the set of products given by the multiplication in $T(Q)$.

Proposition 2.6. [6, Proposition 2.7] *Let Q_1 and Q_2 be finitely generated abelian pro-p groups and π_1 and π_2 be the projections from $Q = Q_1 \oplus Q_2$ to Q_1 and Q_2, respectively. Let A_1 and A_2 be finitely generated pro-p $\mathbb{Z}_p[\![Q_1]\!]$- and $\mathbb{Z}_p[\![Q_2]\!]$-modules, respectively. Then the completed tensor product $A_1 \hat{\otimes}_{\mathbb{Z}_p} A_2$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module and*

$$\Delta_{A_1 \hat{\otimes}_{\mathbb{Z}_p} A_2}(Q) = \big(\pi_1^* \Delta_{A_1}(Q_1)\big)\big(\pi_2^* \Delta_{A_2}(Q_2)\big).$$

3. Homology of pro-p modules

A pro-p group G is of type FP_m over $\mathbb{Z}_p[\![G]\!]$ if $\mathbb{Z}_p$ considered as trivial $\mathbb{Z}_p[\![G]\!]$-module is of type FP_m over $\mathbb{Z}_p[\![G]\!]$. In [6, Theorem A], King proved that G is of type FP_m if and only if each homology group $H_i(G, \mathbb{F}_p) = \mathrm{Tor}_i^{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, \mathbb{F}_p)$ is finite for $i \leqslant m$. Indeed, the result in [6] is more general and it uses the fact that a pro-p $\mathbb{Z}_p[\![G]\!]$-module M is topologically (or abstractly) finitely generated if and only if $M \hat{\otimes}_{\mathbb{Z}_p[\![G]\!]} \mathbb{F}_p$ is finite-dimensional over $\mathbb{F}_p$ [3, Corollary 1.5].

Let N be a (closed) normal subgroup of a pro-p group G. Note that if G/N has finite rank then $\mathbb{Z}_p[\![G/N]\!]$ is topologically (and abstractly) Noetherian [14, Theorem 8.7.8] and G is of type FP_m if and only if $H_i(N, \mathbb{F}_p)$ is a finitely generated pro-p $\mathbb{Z}_p[\![G/N]\!]$-module for $i \leqslant m$ [5, Theorem 3.2].

In this section, we present similar results for the case of non-trivial modules. In what follows, k will denote either $\mathbb{Z}_p$ or $\mathbb{F}_p$.

Theorem 3.1. *Let G be a pro-p group, N a normal subgroup of G such that G/N has finite rank and B a pro-p $\mathbb{Z}_p[\![G]\!]$-module. Then B is of type FP_m over $\mathbb{Z}_p[\![G]\!]$ if and only if $\mathrm{Tor}_i^{\mathbb{Z}_p[\![G]\!]}(B, k[\![G/N]\!])$ is a finitely generated (right) pro-p $k[\![G/N]\!]$-module for $0 \leqslant i \leqslant m$.*

Proof. The proof is a straightforward adaptation of the case $B = \mathbb{Z}_p$ with trivial G-action proved in [5, Theorem A]. We omit the details. □

Lemma 3.2. *Let G be a pro-p group, N a normal subgroup of G and B pro-p $\mathbb{Z}_p[\![G]\!]$-module. Then $\mathrm{Tor}_i^{\mathbb{Z}_p[\![N]\!]}(B, k)$ is a (right) pro-p $k[\![G/N]\!]$-module isomorphic to $\mathrm{Tor}_i^{\mathbb{Z}_p[\![G]\!]}(B, k[\![G/N]\!])$, for all i.*

The above lemma is a pro-p version of the Shapiro's lemma [12, Theorem 6.10.9]. In view of Lemma 3.2 and Theorem 3.1, we can state the following result which is the main tool to be used in the proof of Theorem 1.1.

Theorem 3.3. *Let G be a pro-p group, N a normal subgroup of G such that G/N has finite rank and B a pro-p $\mathbb{Z}_p[\![G]\!]$-module. Then B is of type FP_m over $\mathbb{Z}_p[\![G]\!]$ if and only if $\mathrm{Tor}_i^{\mathbb{Z}_p[\![N]\!]}(B, k)$ is finitely generated as a pro-p $k[\![G/N]\!]$-module for all $0 \leqslant i \leqslant m$.*

4. Proof of Theorem 1.1

We start this section by proving that (ii) is equivalent to (iv) in Theorem 1.1. Indeed, this is a particular case of the following result.

Theorem 4.1. *Let Q be a finitely generated abelian pro-p group and $A_1, A_2, \ldots, A_m$ finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-modules. Then the completed tensor product $C = A_1 \hat{\otimes}_{\mathbb{Z}_p} A_2 \hat{\otimes}_{\mathbb{Z}_p} \cdots \hat{\otimes}_{\mathbb{Z}_p} A_m$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action if and only if whenever $v_i \in \Delta_{A_i}(Q)$, $i = 1, 2, \ldots, m$, are such that $v_1 v_2 \cdots v_m = 1$ we have $v_i = 1$, for $i = 1, 2, \ldots, m$.*

Proof. Since A_i is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module for $i = 1, 2, \ldots, m$, we have that C is finitely generated as $\hat{\bigotimes}^m_{\mathbb{Z}_p} \mathbb{Z}_p[\![Q]\!] (\cong \mathbb{Z}_p[\![Q^m]\!])$-module.

Let $\delta : Q \to Q^m$, $q \mapsto (q, q, \ldots, q)$, be the diagonal embedding. By Proposition 2.5, C is finitely generated as a $\mathbb{Z}_p[\![\delta(Q)]\!]$-module if and only if

$$\Delta_C\big(Q^m\big) \cap T\big(Q^m, \delta(Q)\big) = \{1\}.$$

Let $\pi_i : Q^m \to Q$ be the ith projection and $\pi_i^* : T(Q) \to T(Q^m)$ be the induced monomorphism. Let us identify $T(Q)$ with its images $\pi_i^*(T(Q))$. We may write each element $v \in T(Q^m)$ uniquely as $v = v_1 v_2 \cdots v_m$ with $v_i \in \pi_i^*(T(Q))$, $i = 1, 2, \ldots, m$. Moreover, we have

$$\bar{v}(x_1 \mathbin{\hat{\otimes}} x_2 \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} x_m) = \bar{v}_1(x_1)\bar{v}_2(x_2)\cdots\bar{v}_m(x_m).$$

By Proposition 2.6,

$$\Delta_C\big(Q^m\big) = \big(\pi_1^*\Delta_{A_1}(Q)\big)\big(\pi_2^*\Delta_{A_2}(Q)\big)\cdots\big(\pi_m^*\Delta_{A_m}(Q)\big).$$

Thus $A_1 \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} A_2 \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \cdots \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} A_m$ is finitely generated as a pro-p $\mathbb{Z}_p[\![\delta(Q)]\!]$-module if and only if

$$\big[\big(\pi_1^*\Delta_{A_1}(Q)\big)\big(\pi_2^*\Delta_{A_2}(Q)\big)\cdots\big(\pi_m^*\Delta_{A_m}(Q)\big)\big] \cap T\big(Q^m, \delta(Q)\big) = 1.$$

But $v \in T(Q^m, \delta(Q))$ if and only if $v_1(q)v_2(q)\cdots v_m(q) = 1$, for all $q \in Q$. Also $v \in (\pi_1^*\Delta_{A_1}(Q))(\pi_2^*\Delta_{A_2}(Q))\cdots(\pi_m^*\Delta_{A_m}(Q))$ if and only if $v_i \in \Delta_{A_i}(Q)$, for $i = 1, 2, \ldots, m$. Moreover, $v = 1$ if and only if $v_i = 1$ for all i. These are precisely the conditions that appear in the statement of the theorem. □

Hereafter we will use the following notation. Let $1 \to A \to G \to Q \to 1$ be an exact sequence of pro-p groups such that A and Q are abelian and G is finitely generated and let B be a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module. Then B is a pro-p $\mathbb{Z}_p[\![G]\!]$-module via the epimorphism $G \to Q$ and A is a finitely generated right pro-p $\mathbb{Z}_p[\![Q]\!]$-module where the action of Q is via conjugation in G.

Theorem 4.2. *If $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action, then $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (\hat{\bigotimes}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action.*

Proof. By Theorem 4.1, $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (\hat{\bigotimes}^m_{\mathbb{Z}_p} A)$ is finitely generated via the diagonal Q-action if and only if whenever $v_1 \in \Delta_B(Q)$ and $v_2, \ldots, v_{m+1} \in \Delta_A(Q)$ are such that $v_1 v_2 \cdots v_{m+1} = 1$ we have $v_1 = v_2 = \cdots = v_{m+1} = 1$.

Suppose that the theorem is false. Then there exist elements $v_1 \in \Delta_B(Q)$ and $v_2, \ldots, v_{m+1} \in \Delta_A(Q)$ not all trivial such that $v_1 v_2 \cdots v_{m+1} = 1$. If $v_1 = 1$, we have m elements in $\Delta_A(Q)$ not all trivial with product equal to 1. Then $\hat{\bigwedge}^m_{\mathbb{Z}_p} A$ is not finitely generated over $\mathbb{Z}_p[\![Q]\!]$, since Theorem 1.1 holds for $B = \mathbb{Z}_p$ [8, Theorem D]. On the other hand, let J be the unique maximal ideal in $\mathbb{Z}_p[\![Q]\!]$ and $B_0 = B/BJ$. By the topological

Nakayama's lemma, B_0 is not zero and, since B is finitely generated over $\mathbb{Z}_p[\![Q]\!]$, by [3, Corollary 1.5], B_0 is finite-dimensional over $\mathbb{Z}_p[\![Q]\!]/J \cong \mathbb{F}_p$. Now, $B_0 \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is a $\mathbb{Z}_p[\![Q]\!]$-module quotient of $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ and therefore is finitely generated over $\mathbb{Z}_p[\![Q]\!]$. Since Q acts trivially on B_0, we have

$$B_0 \hat{\otimes}_{\mathbb{Z}_p} \left(\hat{\bigwedge}^m_{\mathbb{Z}_p} A\right) \cong \bigoplus_d \left(\mathbb{F}_p \hat{\otimes}_{\mathbb{Z}_p} \left(\hat{\bigwedge}^m_{\mathbb{Z}_p} A\right)\right)$$

as pro-p $\mathbb{Z}_p[\![Q]\!]$-modules, where $d = \dim_{\mathbb{F}_p} B_0$. It follows that $\mathbb{F}_p \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is finitely generated over $\mathbb{Z}_p[\![Q]\!]$ and therefore the same holds for $\hat{\bigwedge}^m_{\mathbb{Z}_p} A$, a contradiction. Thus we may suppose that $v_1 \neq 1$.

In order to get a contradiction we follow some ideas from [8, Theorem A]. Firstly, we will reduce the problem to the case where A and B are cyclic pro-p $\mathbb{Z}_p[\![Q]\!]$-modules with characteristic p. By [6, Lemmas 2.3 and 2.4], if M is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module we have

$$\Delta_M(Q) = \Delta_{\frac{M}{pM}}(Q) = \Delta_{\frac{\mathbb{Z}_p[\![Q]\!]}{M^\circ}}(Q) = \Delta_{\frac{\mathbb{Z}_p[\![Q]\!]}{\sqrt{M^\circ}}}(Q),$$

where $M^\circ = \mathrm{Ann}_{\mathbb{Z}_p[\![Q]\!]}(M)$. Then we can assume that A and B have prime characteristic p. Since $\mathbb{Z}_p[\![Q]\!]$ is an abstractly Noetherian commutative ring, $\sqrt{A^\circ} = P_1 \cap \cdots \cap P_s$ and $\sqrt{B^\circ} = Q_1 \cap \cdots \cap Q_r$ are intersections of the minimal prime ideals of A° and B°, respectively. Recall that the minimal prime ideals of A° and B° as ideals of $\mathbb{Z}_p[\![Q]\!]$ coincide with the minimal prime ideals of A and B as abstract $\mathbb{Z}_p[\![Q]\!]$-modules, respectively. For each i, choose an element $a_i \in A$ such that $\mathrm{Ann}_{\mathbb{Z}_p[\![Q]\!]}(a_i) = P_i$. By [1, Proposition II.1.4], the map

$$\phi : \bigoplus_{i=1}^{s} \frac{\mathbb{Z}_p[\![Q]\!]}{P_i} \to A$$

that sends $\bigoplus(\lambda_i + P_i)$ to $\sum \lambda_i a_i$ is injective. Since $\tilde{A} := \mathbb{Z}_p[\![Q]\!]/\sqrt{A^\circ}$ embeds in $\bigoplus_{i=1}^s \mathbb{Z}_p[\![Q]\!]/P_i$, it follows that $\tilde{A}$ embeds in A. Now, by [14, Lemma 7.2.2], every finitely generated abstract $\mathbb{Z}_p[\![Q]\!]$-submodule of A is closed, then $\tilde{A}$ embeds in A as a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-submodule. By the same arguments, we have that $\tilde{B} := \mathbb{Z}_p[\![Q]\!]/\sqrt{B^\circ}$ embeds in B as a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-submodule. Now, since we are supposing that A and B have prime characteristic p, $\tilde{B} \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} \tilde{A})$ embeds in $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$, from where it follows that $\tilde{B} \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} \tilde{A})$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action. Therefore, the hypothesis hold for $\tilde{A}$ and $\tilde{B}$ and we can assume $\tilde{A} = A$ and $\tilde{B} = B$.

Recall that we are supposing that there exist elements $v_1 \in \Delta_B(Q)$, with $v_1 \neq 1$, and $v_2, \ldots, v_{m+1} \in \Delta_A(Q)$ not all trivial such that $v_1 v_2 \cdots v_{m+1} = 1$. Without loss of generality, suppose that $v_2, \ldots, v_s$ are non-trivial and $v_{s+1} = \cdots = v_{m+1} = 1$. For $i = 1, 2, \ldots, s$, each v_i extends to an unique non-trivial continuous ring homomorphism $\bar{v}_i : \mathbb{Z}_p[\![Q]\!] \to \mathbb{F}[\![t]\!]$ such that $B^\circ \leqslant \operatorname{Ker} \bar{v}_1$ and $A^\circ \leqslant \operatorname{Ker} \bar{v}_i$, $i = 2, \ldots, s$. Let

$$w_1 : B \to \mathbb{F}[\![t]\!], \qquad w_i : A \to \mathbb{F}[\![t]\!], \quad i = 2, \ldots, s,$$

be the homomorphisms induced by $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_s$, respectively, and $\mu_i : \mathbb{F}[\![t]\!] \to \mathbb{F}[\![t_i]\!]$, $i = 1, 2, \ldots, m+1$, the isomorphisms of $\mathbb{F}$-algebras that send t to t_i.

Since $\mathbb{F}$ has the discrete topology and is infinite, $\mathbb{F}[\![t]\!]$ is not a pro-p ring but does have much in common with pro-p rings. In order to do the "complete tensor product" of the above maps, we will define another tensor product that has the same properties of $\hat{\otimes}$. If K is a field and $M = \varprojlim_k M_k$ and $N = \varprojlim_l N_l$ are vector spaces over K given by the inverse limit of finite-dimensional vector spaces over K, we define $M \bar{\otimes}_K N$ by the inverse limit of abstract tensor products $\varprojlim_{k,l}(M_k \otimes_K N_l)$. Now, we have that $\mathbb{F}[\![t]\!] = \varprojlim_k \mathbb{F}[\![t]\!]/\langle t^k \rangle$ is the inverse limit of finite-dimensional spaces over $\mathbb{F}$, where $\langle t^k \rangle$ is the ideal of $\mathbb{F}[\![t]\!]$ generated by t^k. Then

$$\mathbb{F}[\![t_i]\!] \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_j]\!] = \varprojlim_{k,l} \left(\frac{\mathbb{F}[\![t_i]\!]}{\langle t_i^k \rangle} \otimes_{\mathbb{F}} \frac{\mathbb{F}[\![t_j]\!]}{\langle t_j^l \rangle} \right) = \mathbb{F}[\![t_i, t_j]\!]$$

and we can construct a non-trivial map

$$\tilde{w} = \mu_1 w_1 \bar{\otimes} \mu_2 w_2 \bar{\otimes} \cdots \bar{\otimes} \mu_s w_s \bar{\otimes} \mu_{s+1} w_s \bar{\otimes} \cdots \bar{\otimes} \mu_{m+1} w_s$$

from $B \hat{\otimes}_{\mathbb{F}_p} (\hat{\bigotimes}^m_{\mathbb{F}_p} A)$ to $\mathbb{F}[\![t_1]\!] \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_2]\!] \bar{\otimes}_{\mathbb{F}} \cdots \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_{m+1}]\!]$ such that

$$\tilde{w}(b \hat{\otimes} a_1 \hat{\otimes} \cdots \hat{\otimes} a_m) = \mu_1 w_1(b) \bar{\otimes} \mu_2 w_2(a_1) \bar{\otimes} \cdots \bar{\otimes} \mu_{m+1} w_s(a_m).$$

Let

$$\alpha : B \hat{\otimes}_{\mathbb{F}_p} \left(\hat{\bigotimes}^m_{\mathbb{F}_p} A \right) \to B \hat{\otimes}_{\mathbb{F}_p} \left(\hat{\bigotimes}^m_{\mathbb{F}_p} A \right)$$

be the $\mathbb{F}_p$-linear map given by

$$\alpha(b \hat{\otimes} a_1 \hat{\otimes} \cdots \hat{\otimes} a_m) = \sum_{\sigma \in S_m} (-1)^\sigma b \hat{\otimes} a_{\sigma(1)} \hat{\otimes} \cdots \hat{\otimes} a_{\sigma(m)},$$

where $(-1)^\sigma$ is the sign of $\sigma \in S_m$. Note that the image $\operatorname{Im} \alpha$ of α factors through $B \hat{\otimes}_{\mathbb{F}_p} (\hat{\bigwedge}^m_{\mathbb{F}_p} A)$. Since $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action, the same holds for $\operatorname{Im} \alpha$.

Let S be the additive subgroup of $\hat{\bigotimes}^m_{\mathbb{F}_p} \mathbb{F}_p[\![Q]\!] \cong \mathbb{F}_p[\![Q^m]\!]$ generated by

$$\left\{ \lambda \in \mathbb{F}_p[\![Q^m]\!] \mid \lambda\sigma = \lambda, \text{ for all } \sigma \in S_m \right\},$$

where σ acts on $\lambda_1 \hat{\otimes} \cdots \hat{\otimes} \lambda_m \in \mathbb{F}_p[\![Q^m]\!]$ permuting the factors. Note that S is closed and $\mathbb{F}_p \subset S$. We claim that $\mathbb{F}_p[\![Q^m]\!]$ is integral over S, in the sense that $\mathbb{F}_p[\![Q^m]\!]$ is abstractly finitely generated as a S-module, hence finitely generated as a pro-p S-module [14, Lemma 7.2.2]. Indeed, each element $t \in \mathbb{F}_p[\![Q^m]\!]$ is integral over S, since it is a root of the polynomial $\prod_{\sigma\in S_m}(x - t\sigma) \in S[x]$. Thus, if X_i is a finite set of topological generators for the ith copy of the abelian pro-p group Q in $Q^m \subset \mathbb{F}_p[\![Q^m]\!]$ and $X = \bigcup_{i=1}^m X_i$, we have that the abstract ring $S[X]$ is integral over S and then $S[X]$ is finitely generated as an abstract S-module. Furthermore, $S \subset S[X] \subseteq \mathbb{F}_p[\![Q^m]\!]$, from which follows that $S[X]$ is closed [14, Lemma 7.2.2]. Then $\mathbb{F}_p[\![Q^m]\!] \subseteq \overline{\mathbb{F}_p[X]} \subseteq \overline{S[X]} = S[X]$ and so $\mathbb{F}_p[\![Q^m]\!] = S[X]$ is integral over S, as claimed.

By using the claim, we have that $\hat{\bigotimes}_{\mathbb{F}_p}^{m+1} \mathbb{F}_p[\![Q]\!] \cong \mathbb{F}_p[\![Q]\!] \hat{\otimes}_{\mathbb{F}_p} (\hat{\bigotimes}_{\mathbb{F}_p}^{m} \mathbb{F}_p[\![Q]\!])$ is integral over $\mathbb{F}_p[\![Q]\!] \hat{\otimes}_{\mathbb{F}_p} S$. Note that $\operatorname{Im}\alpha(\mathbb{F}_p[\![Q]\!] \hat{\otimes}_{\mathbb{F}_p} S) = \operatorname{Im}\alpha$ and recall that $\operatorname{Im}\alpha$ is a finitely generated $\mathbb{F}_p[\![Q]\!]$-module via the diagonal Q-action. So $V = \operatorname{Im}\alpha\,(\hat{\bigotimes}_{\mathbb{F}_p}^{m+1} \mathbb{F}_p[\![Q]\!])$ is a finitely generated $\mathbb{F}_p[\![Q]\!]$-module via the diagonal Q-action.

Recall that

$$\begin{aligned} R &:= \mathbb{F}[\![t_1]\!] \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_2]\!] \bar{\otimes}_{\mathbb{F}} \cdots \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_{m+1}]\!] \\ &\cong \mathbb{F}[\![t_1, t_2, \ldots, t_{m+1}]\!] \cong \mathbb{F}[\![t_1, \ldots, t_s]\!] \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_{s+1}, \ldots, t_{m+1}]\!]. \end{aligned}$$

We have $\tilde{w}(V)$ finitely generated over $\mathbb{Z}_p[\![Q]\!]$, where the action of Q on R is induced by $\tilde{w}$ via the diagonal Q-action on $B \hat{\otimes}_{\mathbb{F}_p} (\hat{\bigotimes}_{\mathbb{F}_p}^{m} A)$, that is, the action of $q \in Q$ is the multiplication by

$$\delta(q) = \mu_1 w_1(q) \mu_2 w_2(q) \cdots \mu_s w_s(q) \mu_{s+1} w_s(q) \cdots \mu_{m+1} w_s(q). \tag{4.3}$$

The same arguments from [8, Lemma 1] shows that $\tilde{w}(V) \neq 0$. Then there exists $j \geqslant 1$ such that $\tilde{w}(V)$ is not contained in $I = \mathbb{F}[\![t_1, \ldots, t_s]\!] \bar{\otimes}_{\mathbb{F}} I_0$, where I_0 is the ideal of $\mathbb{F}[\![t_{s+1}, \ldots, t_{m+1}]\!]$ generated by $t_{s+1}^j, \ldots, t_{m+1}^j$. Let J_0 be the ideal of $\mathbb{F}[\![t_1, \ldots, t_s]\!]$ generated by $t_1 - t_2, t_2 - t_3, \ldots, t_{s-1} - t_s$ and set $J = J_0 \bar{\otimes}_{\mathbb{F}} \mathbb{F}[\![t_{s+1}, \ldots, t_{m+1}]\!]$. Since $\bigcap_{k\geqslant 1} J_0^k = 0$ and

$$R/I \cong \mathbb{F}[\![t_1, \ldots, t_s]\!] \bar{\otimes}_{\mathbb{F}} \big(\mathbb{F}[\![t_{s+1}, \ldots, t_{m+1}]\!]/I_0\big),$$

it follows that $\bigcap_{k\geqslant 1}(J^k + I)/I = 0$. Then, since $\tilde{w}(V)$ is not a subset of I, there exists $k_0 \geqslant 0$ such that

$$\tilde{w}(V) \subseteq J^{k_0} + I \quad \text{and} \quad \tilde{w}(V) \nsubseteq J^{k_0+1} + I,$$

where by definition $J^0 = \mathbb{F}[\![t_1, t_2, \ldots, t_{m+1}]\!]$.

Recalling that an element $q \in Q$ acts on $\mathbb{F}[\![t_1, t_2, \ldots, t_{m+1}]\!]$ as in (4.3), that $v_1 v_2 \cdots v_s = 1$ and that the action of Q is continuous, we have

$$\begin{aligned}\delta(q) &\in J + \mu_1\big(w_1(q)\cdots w_s(q)\big)\mu_{s+1}w_s(q)\cdots\mu_{m+1}w_s(q)\\ &= J + \mu_{s+1}w_s(q)\cdots\mu_{m+1}w_s(q)\\ &\subseteq J + 1 + (t_{s+1},\dots,t_{m+1})\mathbb{F}[\![t_{s+1},\dots,t_{m+1}]\!].\end{aligned}$$

Then for sufficiently large k, we have

$$\delta\big(q^{p^k}\big) = \delta(q)^{p^k} \in J + I + 1.$$

Therefore the canonical image $\overline{V}$ of $\tilde{w}(V)$ in

$$\big(J^{k_0}+I\big)/\big(J^{k_0+1}+I\big) \cong \big(J_0^{k_0}/J_0^{k_0+1}\big)\,\hat{\otimes}_{\mathbb{F}}\,\big(\mathbb{F}[\![t_{s+1},\dots,t_{m+1}]\!]/I_0\big)$$

is not trivial and the induced action of Q^{p^k} on $\overline{V}$ is trivial. Since $\tilde{w}(V)$ is finitely generated over $\mathbb{F}_p[\![Q]\!]$ we deduce that $\overline{V}$ is finitely generated over $\mathbb{F}_p(Q/Q^{p^k})$ and so finite.

On the other hand, choose $q \in Q$ such that $\mu_1 v_1(q) \neq 1$ and define

$$h = q\,\hat{\otimes}\left(\hat{\bigotimes}^{m} 1\right) \in \hat{\bigotimes}^{m+1}_{\mathbb{F}_p}\mathbb{F}_p[\![Q]\!].$$

The action of h on $(J^{k_0}+I)/(J^{k_0+1}+I)$ is given by multiplication by $f = \mu_1 v_1(q)$ with $f \neq 1$ in $\mathbb{F}[\![t_1]\!]$. Note that $J_0^{k_0}/J_0^{k_0+1}$ is a free abstract $\mathbb{F}[\![t_1]\!]$-module of finite rank and $\mathbb{F}$ is a domain. It follows that the non-trivial powers of f cannot act trivially on any non-zero element of $(J^{k_0}+I)/(J^{k_0+1}+I)$. But, since the image $\overline{V}$ of $\tilde{w}(V)$ in $(J^{k_0}+I)/(J^{k_0+1}+I)$ is finite, some power of f must act trivially on $\overline{V}$, a contradiction. □

With Theorems 4.1 and 4.2, we have proven the equivalences of (ii), (iii) and (iv) from Theorem 1.1, since (ii) implies (iii) trivially. Furthermore, we have also proven another equivalence for the item (i) in Theorem 3.3. In Lemma 4.5, we will establish a relationship between $\mathrm{Tor}_n^{\mathbb{Z}_p[\![A]\!]}(B,\mathbb{F}_p)$ and the homology group $H_n(A,\mathbb{F}_p)$.

Lemma 4.4. *If M and N are abelian pro-p groups such that N has exponent p, there is a natural isomorphism* $\mathrm{Tor}_1^{\mathbb{Z}_p}(M,N) \cong ({}_pM)\,\hat{\otimes}_{\mathbb{Z}_p} N$, *where* $({}_pM) = \{m \in M \mid pm = 0\}$.

Proof. Firstly note that, since the functors $\mathrm{Tor}_1^{\mathbb{Z}_p}(M,-)$ and $({}_pM)\,\hat{\otimes}_{\mathbb{Z}_p} -$ commute with inverse limits, we can consider N finite. Since $pN = 0$, we have that N is a pro-p $\mathbb{Z}_p/p\mathbb{Z}_p(\cong \mathbb{F}_p)$-module. Let X be a (finite) base of N as a pro-p $\mathbb{F}_p$-module and F be a free pro-p $\mathbb{Z}_p$-module on X. Then $0 \to F \xrightarrow{p} F \to N \to 0$ is a $\mathbb{Z}_p$-free resolution of N, where $F \xrightarrow{p} F$ is the multiplication by p and $F \to N$ is the identity on X. Therefore

$$\mathrm{Tor}_1^{\mathbb{Z}_p}(M,N) = \mathrm{Ker}\big(M\,\hat{\otimes}_{\mathbb{Z}_p} F \xrightarrow{1\hat{\otimes} p} M\,\hat{\otimes}_{\mathbb{Z}_p} F\big).$$

Since $F \cong \bigoplus_{x\in X} \mathbb{Z}_p x$, we have $M \hat{\otimes}_{\mathbb{Z}_p} F \cong \bigoplus_{x\in X} Mx$, where $Mx \cong M$ as a pro-p $\mathbb{Z}_p$-module. Thus,

$$\operatorname{Ker}\left(\bigoplus_{x\in X} Mx \xrightarrow{p} \bigoplus_{x\in X} Mx\right) \cong \bigoplus_{x\in X} ({}_pM)x \cong \frac{({}_pM)\hat{\otimes}_{\mathbb{Z}_p} F}{({}_pM)\hat{\otimes}_{\mathbb{Z}_p} pF} \cong ({}_pM)\hat{\otimes}_{\mathbb{Z}_p} N$$

and the lemma is proved. □

Recalling that by hypothesis A acts trivially on B, we can prove the following version of the universal coefficient theorem.

Lemma 4.5. *There is a short exact sequence of $\mathbb{Z}_p[\![Q]\!]$-modules*

$$0 \to B \hat{\otimes}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p) \to \operatorname{Tor}_n^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p) \to ({}_pB)\hat{\otimes}_{\mathbb{Z}_p} H_{n-1}(A, \mathbb{F}_p) \to 0$$

with natural maps.

Proof. Recall that $\operatorname{Tor}_n^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$ can be calculated starting from a $\mathbb{Z}_p[\![A]\!]$-projective resolution of B or starting from a $\mathbb{Z}_p[\![A]\!]$-projective resolution of $\mathbb{F}_p$. Let

$$\mathcal{F}: \cdots \to F_i \to F_{i-1} \to \cdots \to F_0 \to \mathbb{F}_p \to 0$$

be a $\mathbb{Z}_p[\![A]\!]$-free resolution of $\mathbb{F}_p$. Then we have that

$$H_n\big(B \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}\big) = \operatorname{Tor}_n^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p).$$

Note also that by hypothesis B is a trivial A-module, so $B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} -$ and $B \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} -$ are isomorphic functors.

By the universal coefficient theorem for pro-p homology (for ordinary homology see [13, Theorem 8.22]), we have that

$$\begin{aligned} 0 &\to B \hat{\otimes}_{\mathbb{Z}_p} H_n\big(\mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}\big) \to H_n\big(B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}\big) \\ &\to \operatorname{Tor}_1^{\mathbb{Z}_p}\big(B, H_{n-1}\big(\mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}\big)\big) \to 0 \end{aligned}$$

is an exact sequence with natural maps. From the above observations we see that the middle term of the exact sequence is $\operatorname{Tor}_n^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$. Moreover, the left term is

$$B \hat{\otimes}_{\mathbb{Z}_p} H_n\big(\mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}\big) = B \hat{\otimes}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p)$$

and, by Lemma 4.4, the right term $\operatorname{Tor}_1^{\mathbb{Z}_p}(B, H_{n-1}(\mathbb{Z}_p \hat{\otimes}_{\mathbb{Z}_p[\![A]\!]} \mathcal{F}_{\mathbb{F}_p}))$ is equal to

$$\operatorname{Tor}_1^{\mathbb{Z}_p}\big(B, H_{n-1}(A, \mathbb{F}_p)\big) \cong ({}_pB)\hat{\otimes}_{\mathbb{Z}_p} H_{n-1}(A, \mathbb{F}_p),$$

since $H_{n-1}(A, \mathbb{F}_p)$ has exponent p. □

We observe that, since $H_i(A, \mathbb{F}_p)$ has exponent p, we have

$$B \hat{\otimes}_{\mathbb{Z}_p} H_i(A, \mathbb{F}_p) \cong (B/pB) \hat{\otimes}_{\mathbb{F}_p} H_i(A, \mathbb{F}_p).$$

So, reducing modulo p will be a natural procedure for the proof of the next results. We will do this many times inside our next proofs, although another kind of reduction could be done. Applying Lemmas 4.5 and 4.8, we can prove that $\mathrm{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$, for $i \leqslant n$, are finitely generated over $\mathbb{Z}_p[\![Q]\!]$ if and only if $\mathrm{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B/pB, \mathbb{F}_p)$ and $\mathrm{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(({}_pB), \mathbb{F}_p)$, for $i \leqslant n$, are finitely generated over $\mathbb{Z}_p[\![Q]\!]$. Thus, by Theorem 3.3, we have that B is of type FP_m over $\mathbb{Z}_p[\![G]\!]$ if and only if $B/pB \cong B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p$ and $({}_pB)$ are of type FP_m over $\mathbb{Z}_p[\![G]\!]$.

Now we are ready to prove that (i) implies (iii).

Theorem 4.6. *If B is of type FP_m over $\mathbb{Z}_p[\![G]\!]$, then $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action.*

Proof. By Theorem 3.3, we have that $\mathrm{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module for $0 \leqslant i \leqslant m$. By Lemma 4.5 and the fact that Q has finite rank, we have $B \hat{\otimes}_{\mathbb{Z}_p} H_i(A, \mathbb{F}_p)$ finitely generated via the diagonal Q-action for $0 \leqslant i \leqslant m$ and therefore $(B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p) \hat{\otimes}_{\mathbb{F}_p} H_m(A, \mathbb{F}_p)$ is finitely generated as a pro-p $\mathbb{F}_p[\![Q]\!]$-module, where the Q-action is the diagonal one. Furthermore, from Theorem 2.1 and the exactness of $- \hat{\otimes}_{\mathbb{F}_p} -$ it follows that

$$1 \hat{\otimes} \beta : (B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p) \hat{\otimes}_{\mathbb{F}_p} \left(\hat{\bigwedge}^m_{\mathbb{F}_p} (A \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p) \right) \to \big(B \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p \big) \hat{\otimes}_{\mathbb{F}_p} H_m(A, \mathbb{F}_p)$$

is a natural monomorphism of pro-p $[\![\mathbb{F}_p Q]\!]$-modules via the diagonal Q-action. Then $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{F}_p} A \hat{\otimes}_{\mathbb{Z}_p} \mathbb{F}_p)$ is finitely generated via the diagonal Q-action and we conclude that $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigwedge}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action. □

It remains to show that (ii) implies (i). This will be done by using Lemma 4.5 and the filtrations of the homology group $H_n(A, \mathbb{F}_p)$ given by Theorem 2.2 and Corollary 2.3. So we need the following lemma.

Lemma 4.7. *Let M and N be pro-p $\mathbb{Z}_p[\![Q]\!]$-modules and*

$$0 \subsetneq F_1 \subseteq F_2 \subseteq \cdots \subsetneq F_s = N$$

be a filtration of $\mathbb{Z}_p[\![Q]\!]$-submodules of N. Then $M \hat{\otimes}_{\mathbb{Z}_p} N$ has a filtration of $\mathbb{Z}_p[\![Q]\!]$-submodules

$$0 \subsetneq Q_1 \subseteq Q_2 \subseteq \cdots \subsetneq Q_s = M \hat{\otimes}_{\mathbb{Z}_p} N,$$

where $Q_j = \mathrm{Im}(M \hat{\otimes}_{\mathbb{Z}_p} F_j \to M \hat{\otimes}_{\mathbb{Z}_p} N)$ *is the image of the induced map by the inclusion* $F_j \hookrightarrow N$. *Moreover,* $M \hat{\otimes}_{\mathbb{Z}_p} (F_j/F_{j-1})$ *maps surjectively to* Q_j/Q_{j-1} *for each* $j = 1, 2, \ldots, s$.

Proof. The first part of the lemma follows immediately from the definitions of Q_j. In order to prove the second part we apply induction on the length s of the filtration of N. The case $s = 1$ is trivial. Suppose that $s > 1$. Then, if $Q'_j = \mathrm{Im}(M \hat{\otimes}_{\mathbb{Z}_p} F_j \to M \hat{\otimes}_{\mathbb{Z}_p} F_{s-1})$, $j = 1, \ldots, s-1$, by induction

$$M \hat{\otimes}_{\mathbb{Z}_p} (F_j/F_{j-1}) \twoheadrightarrow Q'_j/Q'_{j-1},$$

where we are using the symbol "$\twoheadrightarrow$" to denote a surjective map. But $Q'_j \twoheadrightarrow \mathrm{Im}(M \hat{\otimes}_{\mathbb{Z}_p} F_j \to M \hat{\otimes}_{\mathbb{Z}_p} F_s) = Q_j$, for $j = 1, \ldots, s-1$. Then

$$M \hat{\otimes}_{\mathbb{Z}_p} (F_j/F_{j-1}) \twoheadrightarrow Q_j/Q_{j-1} \quad \text{for } j = 1, \ldots, s-1.$$

Furthermore,

$$M \hat{\otimes}_{\mathbb{Z}_p} (F_s/F_{s-1}) \cong \big(M \hat{\otimes}_{\mathbb{Z}_p} F_s\big) \big/ \mathrm{Im}\big(M \hat{\otimes}_{\mathbb{Z}_p} F_{s-1} \to M \hat{\otimes}_{\mathbb{Z}_p} F_s\big) = Q_s/Q_{s-1},$$

since $B \hat{\otimes}_{\mathbb{Z}_p} -$ is right exact. $\square$

The last auxiliary result is about finitely generation of completed tensor product of pro-p modules.

Lemma 4.8. [8, Lemma 2] *Let* Q *be a pro-*p *group of finite rank. Suppose that* M *and* N *are finitely generated pro-*p $\mathbb{Z}_p[\![Q]\!]$*-modules and that* $M \hat{\otimes}_{\mathbb{Z}_p} N$ *is a finitely generated pro-*p $\mathbb{Z}_p[\![Q]\!]$*-module via the diagonal* Q*-action. Then for every pro-*p $\mathbb{Z}_p[\![Q]\!]$*-submodule* M_1 *of* M*, the tensor product* $M_1 \hat{\otimes}_{\mathbb{Z}_p} N$ *is finitely generated as a pro-*p $\mathbb{Z}_p[\![Q]\!]$*-module via the diagonal* Q*-action.*

We observe that the equivalence of (ii) and (iv) of Theorem 1.1 implies that $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigotimes}^m_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action if and only if $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigotimes}^n_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action for all $n \leqslant m$. Then the next theorem concludes the proof of Theorem 1.1.

Theorem 4.9. *If* $B \hat{\otimes}_{\mathbb{Z}_p} (\hat{\bigotimes}^n_{\mathbb{Z}_p} A)$ *is a finitely generated pro-*p $\mathbb{Z}_p[\![Q]\!]$*-module via the diagonal* Q*-action for all* $n \leqslant m$*, then* B *is of type* FP_m *over* $\mathbb{Z}_p[\![G]\!]$.

Proof. Since Q has finite rank, Theorem 3.3 says that we must show that $\mathrm{Tor}_i^{\mathbb{Z}_p[\![A]\!]}(B, \mathbb{F}_p)$ is finitely generated as a pro-p $\mathbb{F}_p[\![Q]\!]$-module for all $i \leqslant m$. By Lemma 4.5, it is enough to show that $B \hat{\otimes}_{\mathbb{Z}_p} H_i(A, \mathbb{F}_p)$ and $({}_pB) \hat{\otimes}_{\mathbb{Z}_p} H_{i-1}(A, \mathbb{F}_p)$ are finitely generated pro-p $\mathbb{F}_p[\![Q]\!]$-modules via the diagonal Q-action for all $i \leqslant m$.

Corollary 2.3 gives us a filtration $\{F_j\}$ of $H_n(A, \mathbb{F}_p)$ such that F_j/F_{j-1} is $\mathbb{Z}_p[\![Q]\!]$-subquotient of

$$M^n_{(\alpha_1,\dots,\alpha_s)} = H_{\alpha_1}(A_1, \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_{\alpha_2}(A_2, \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \cdots \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_{\alpha_s}(A_s, \mathbb{F}_p),$$

where $\alpha_1 + \cdots + \alpha_s = n$. Then, by Lemma 4.7, we obtain a filtration $\{Q_j\}$ of $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p)$ such that

$$B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (F_j/F_{j-1}) \twoheadrightarrow Q_j/Q_{j-1}.$$

Note that if Q_j/Q_{j-1} is a finitely generated $\mathbb{F}_p[\![Q]\!]$-module for all j, then $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p)$ is a finitely generated pro-p $\mathbb{F}_p[\![Q]\!]$-module, since $Q_0 = 0$ and $Q_t = B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p)$ for some t. Thus we need to show that $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (F_j/F_{j-1})$ is finitely generated. Since each F_j/F_{j-1} is a subquotient of $M^n_{(\alpha_1,\dots,\alpha_s)}$, it is enough to show that $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} M^n_{(\alpha_1,\dots,\alpha_s)}$ is a finitely generated $\mathbb{F}_p[\![Q]\!]$-module.

By Theorem 2.2, $M^n_{(\alpha_1,\dots,\alpha_s)}$ has a filtration of pro-p $\mathbb{Z}_p[\![Q]\!]$-submodules with factors embeddable in

$$N^n_{(k_1,\dots,k_s)} = \left(\left(\hat{\bigwedge}\nolimits_{\mathbb{F}_p}^{\alpha_1-2k_1} A_1\right) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} \hat{S}^{k_1}_{\mathbb{F}_p}(A_1)\right) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} \cdots \mathbin{\hat{\otimes}}_{\mathbb{F}_p} \left(\left(\hat{\bigwedge}\nolimits_{\mathbb{F}_p}^{\alpha_s-2k_s} A_s\right) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} \hat{S}^{k_s}_{\mathbb{F}_p}(A_s)\right),$$

for some $k_i \leqslant [\alpha_i/2]$. By Lemma 4.7, the filtration of $M^n_{(\alpha_1,\dots,\alpha_s)}$ induces a filtration of $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} M^n_{(\alpha_1,\dots,\alpha_s)}$ with factors that are surjective images of $\mathbb{Z}_p[\![Q]\!]$-modules embeddable in $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} N^n_{(k_1,\dots,k_s)}$. This last module is a surjective image of $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} V$, where

$$V = \left(\hat{\bigotimes}\nolimits_{\mathbb{Z}_p}^{\alpha_1-k_1} A_1\right) \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \left(\hat{\bigotimes}\nolimits_{\mathbb{Z}_p}^{\alpha_2-k_2} A_2\right) \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \cdots \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \left(\hat{\bigotimes}\nolimits_{\mathbb{Z}_p}^{\alpha_s-k_s} A_s\right).$$

By hypothesis, $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} (\hat{\bigotimes}^n_{\mathbb{Z}_p} A) \cong B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (\hat{\bigotimes}^n_{\mathbb{Z}_p} A)$ is finitely generated via the diagonal Q-action for $n \leqslant m$. By Theorem 4.1 together with the fact that Theorem 1.1 holds for $B = \mathbb{Z}_p$, we see that $\hat{\bigotimes}^n_{\mathbb{Z}_p} A$ is finitely generated as a pro-p $\mathbb{Z}_p[\![Q]\!]$-module. Since $\sum_{i=1}^s (\alpha_i - k_i) \leqslant n$, by Lemma 4.8, $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} V$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action. Thus we conclude that $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} N^n_{(k_1,\dots,k_s)}$, is finitely generated via the diagonal Q-action for $n \leqslant m$. Therefore $(B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} \mathbb{F}_p) \mathbin{\hat{\otimes}}_{\mathbb{F}_p} M^n_{(\alpha_1,\dots,\alpha_s)}$ and, consequently, $B \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} H_n(A, \mathbb{F}_p)$ are finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-modules via the diagonal Q-action for $n \leqslant m$.

Observe that, by Lemma 4.8, $({}_pB) \mathbin{\hat{\otimes}}_{\mathbb{Z}_p} (\hat{\bigotimes}^n_{\mathbb{Z}_p} A)$ is a finitely generated pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action for all $n \leqslant m$. Then similar arguments to the above hold if we use $({}_pB)\mathbin{\hat{\otimes}}_{\mathbb{Z}_p}(\hat{\bigotimes}^{n-1}_{\mathbb{Z}_p}A)$ instead $B\mathbin{\hat{\otimes}}_{\mathbb{Z}_p}(\hat{\bigotimes}^{n}_{\mathbb{Z}_p}A)$. Thus $({}_pB)\mathbin{\hat{\otimes}}_{\mathbb{Z}_p}H_{n-1}(A,\mathbb{F}_p)$ is finitely generated as a pro-p $\mathbb{Z}_p[\![Q]\!]$-module via the diagonal Q-action for all $n \leqslant m$, which concludes the proof of the theorem. □

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